Lebanese American University
Department of Computer Science and Mathematics

MTH207 – Discrete Structures 1

Exam 1 – Fall 2014

Exam Duration: 75 minutes

Name: **ID#:**

Instructions: This exam consists of 9 problems and a total of 8 pages (last page is for scratch). Make sure no problem/page is missing. Answer the questions in the space provided for each problem. If more space is needed, you can use the back of the pages. To receive full credits, you have to justify your answers.

Question Number	Grade
1. (10%)	06
2. (8%)	08
3. (12%)	12
4. (10%)	10
5. (10%)	10
6. (12%)	12
7. (15%)	15
8. (15%)	15
9. (8%)	08
TOTAL	96

1. (10%) Show that m is a multiple of 3 if and only if m^2 is a multiple of 3.

show $m=3k \Longleftrightarrow m^2=3k$

first show: $m=3k \Rightarrow m^2=3k$

let $m=3k$; $m^2=3(3k^2)$ ✓

~~let $m^2=3k$~~, second, show: $m^2=3k \Longrightarrow m=3k$

~~let $m^2=3k$~~

contra-positive: $\neg(m=3k) \Longrightarrow \neg(m^2=3k)$

by contrapositive let $m \neq 3k$

$\cancel{m=3k \vee \neg(m^2=3k)}$

$\cancel{(m=3k) \vee \neg(m^2=3k)}$

$\Rightarrow m^2 \neq 3^2k^2$

$\neq 3(3k^2)$

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$\therefore m \neq 3k \Longrightarrow m^2 \neq 3(3k^2)$ True.

2. (8%) Show that $\sqrt{3}$ is an irrational number.

✓

if $\sqrt{3}$ is rational then it can be written as $\frac{m}{n}$ where m & n are two ~~natural~~ integers $(n>0)$, $\frac{m}{n}$ in simplest form (no common divisors).

$\sqrt{3}=\frac{m}{n} \Rightarrow 3=\frac{m^2}{n^2} \Rightarrow 3n^2=m^2 \Rightarrow m^2$ is a multiple of $3 \Rightarrow m^2=3k$, $m=3k$

$\cancel{\Rightarrow n^2=1 \Rightarrow n=1}$

$\Longrightarrow$

m & n have common divisor.

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$\therefore$ first condition is wrong

3. (12%) Show that $((p \vee q) \wedge -p) \rightarrow q$ is a tautology using two different methods.

1) Truth Table

p	q	$-p$	$p \vee q$	$(p \vee q) \wedge -p$	$(p\vee q) \wedge -p \rightarrow q$
T	T	F	T	F	T
F	F	T	F	F	T
T	F	F	T	F	T
F	T	T	T	T	T

(6)

2) $(p \vee q) \wedge -p = \cancel{(p \vee -p) \wedge (q \vee p) \rightarrow q}$

$= \cancel{(p \wedge -p) \vee (p \wedge q) \rightarrow q}$

$\cancel{((p \vee q) \wedge -p) = (p \vee -p) \wedge (q \vee p)} \rightarrow q$

$= \cancel{q \vee p \rightarrow q}$

$= \cancel{-(q \vee p)} \vee q$

$= \cancel{(-q \vee q)} \vee -p$

$= \cancel{(-q \wedge -p) \vee q}$

$= (-q \vee q) \wedge (-p \vee q)$

$= T \wedge$

2

$((p \vee q) \wedge -p) = (\overbrace{p \wedge -p}^{F}) \vee (p \wedge q) \rightarrow q$

$= (p \wedge q) \rightarrow q$

$= -(p \wedge q) \vee q$

$= -p \vee (\underbrace{-q \vee q}_{})$

$= -p \vee T$

$= T$ ✓

(6)

4. **(10%)** Show by induction that $1+4+4^2+....+4^n = \frac{4^{n+1}-1}{3}$ where n is a non-negative integer $(n=0,1,2,...)$

- try for $n=0$; $1 = \frac{4-1}{3} = 3$
- assume $1+4+4^2+\dots+4^n = \frac{4^{n+1}-1}{3}$ is True. $(n=0,1,2,..)$
- show $\underbrace{1+4+4^2+\dots+4^n}_{\frac{4^{n+1}-1}{3}}+4^{n+1} = \frac{4^{n+2}-1}{3}$

$$4^{n+1} = \frac{4^{n+2}-1}{3} - \frac{4^{n+1}-1}{3} \overset{R.H.}{=} \frac{4^{n+1}(4-1)}{3} = 4^{n+1} \quad \text{True}$$

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5. **(10%)** Show by induction that $2^{2n}-1$ is divisible by 3 for any positive integer $n \geq 1$.

- Try for $n=1$: $2^2-1 = 4-1 = 3$
- Assume $2^{2n}-1 = 3k$ for any +ve integer $n \geq 1$
- show $2^{2n+2}-1 = 3k$ for any +ve integer $n \geq 1$ on back

$\cancel{2 \cdot 2^n - 1 \overset{?}{=} 3k;\ 2^n = 3k+1}$

$\cancel{2(3k+1)-1 = 6k+1} =$

$\cancel{2 \cdot 2^n - 1 = 2(2^n - \tfrac{1}{2})} =$

$\cancel{2 \cdot 2^n - 1 + 4 - 4} =$

$\cancel{2(\underbrace{2^n-1}_{3k}) - 2}$

$\cancel{2(3k-1)}$

$$2^{n+1}-1 = 2(\underbrace{2^n-1}_{3k})-3$$
$$= 3k(2-1)$$
$$= 3k \quad \text{True}$$

$\cancel{2^{n+1}-1} = 2(2^n-1)-1$

$= 2(3k)-1$

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$2^{2n+2}-1 = 3k$

$(2^2 \cdot 2^{2n}) - 1 = 4(2^{2n}-1)-3$ ✓

$= 4(3k)-3$ ✓

$= 3(4k)-3$

$= 3(4k-1)$ True ✓

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6. Recall that $n! = 1.2.3....n$; for example $1! = 1; 2! = 2; 3! = 6; 4! = 24; 5! = 120; 6! = 720..etc...$

Let $P(n)$ represent the statement: $P(n): 1.1!+2.2!+3.3!+...+n.n!$

a. **(4%)** Calculate $P(1), P(2), P(3), P(4)$

$P(1) = 1$

$P(2) = 5$

$P(3) = 1+4+18 = 23$

$P(4) = \cancel{23+24=47}$

$= 23 + 4(24) = 23 + 96 = 119$

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b. **(2%)** Conjecture a formula for $P(n)$

$P(n) = (n+1)! - 1$

2

c. **(6%)** Prove your conjecture by mathematical induction.

- try for $n=1$: $P(1) = 2! - 1 = 1$
- assume $P(n) = (n+1)! - 1 = 1.1! + 2.2! + 3.3! + \dots + n.n!$
- show $P(n+1) = (n+2)! - 1 = \underbrace{1.1! + 2.2! + 3.3! + \dots + n.n!}_{(n+1)! - 1} + (n+1)(n+1)!$

$+ (n+1)(n+1)!$

$RHS = (n+1)!(1 + n+1) - 1$

$= \underbrace{(n+1)!\,(n+2)}{} - 1$

$= (n+2)! - 1$

True.

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7. Given three sets A, B, and C; re-write the following using predicate logic:

a. **(5%)** $A \subseteq B \cup C$

$\forall x\,(x \in A \longrightarrow (x \in B) \vee (x \in C))]$

5

b. **(5%)** $A \subset B$

$\forall x\,[(x \in A \rightarrow x \in B) \wedge \neg(x \in B \longrightarrow x \in A)]$

5

c. **(5%)** $A = B$

$\forall x\,(x \in A \longleftrightarrow x \in B)$

5

8. Let *s* denote the students at LAU, *c* the courses offered at LAU, *d* the departments at LAU, and *P(s,c,d)* denote that student *s* has taken course *c* in department *d*.

a. **(5%)** Express the following using quantifiers: *There is a student who has taken some course in every department at LAU.*

$\exists s\,\forall d\,\exists c\; P(s,c,d)$

no, this is correct.

5

b. **(5%)** Using quantifiers, write the negation of the one you found in part a.

$\forall s\,\exists d\,\forall c\; \neg P(s,c,d)$

5

c. **(5%)** Express the negation in part b using English.

All students have not taken a course in some department at LAU

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9. (8%) Let A, B, and C denote three sets. Show the following: If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then $B = C$.

if $B = C \Rightarrow B \subseteq C$ and $C \subseteq B$

1) $B \subseteq C$

- $x \in B$

if $x \in A$; since $A \cap B = A \cap C$

$\longrightarrow x \in C$

if $x \notin A$; since $A \cup B = A \cup C$

$\longrightarrow x \in C$

$C \subseteq B$

- $x \in C$

if $x \in A$; since $A \cap B = A \cap C$

$\longrightarrow x \in B$

if $x \notin A$ since $A \cup B = A \cup C$

$\longrightarrow x \in C$

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